

**What Is Claimed Is:**

- 1           1.       A method for using a computer system to solve a global inequality  
2       constrained optimization problem specified by a function  $f$  and a set of inequality  
3       constraints  $p_i(\mathbf{x}) \leq 0$  ( $i=1, \dots, m$ ), wherein  $f$  and  $p_i$  are scalar functions of a vector  
4        $\mathbf{x} = (x_1, x_2, x_3, \dots, x_n)$ , the method comprising:  
5               receiving a representation of the function  $f$  and the set of inequality  
6       constraints at the computer system;  
7               storing the representation in a memory within the computer system;  
8               performing an interval inequality constrained global optimization process  
9       to compute guaranteed bounds on a globally minimum value of the function  $f(\mathbf{x})$   
10       subject to the set of inequality constraints;  
11               wherein performing the interval global optimization process involves,  
12                       applying term consistency to the set of inequality  
13                       constraints over a subbox  $\mathbf{X}$ , and  
14                       excluding any portion of the subbox  $\mathbf{X}$  that is proved to be  
15                       in violation of at least one member of the set of inequality  
16                       constraints.
- 1           2.       The method of claim 1, further comprising:  
2               linearizing the set of inequality constraints to produce a set of linear  
3       inequality constraints with interval coefficients that enclose the nonlinear  
4       constraints;  
5               preconditioning the set of linear inequality constraints through additive  
6       linear combinations to produce a preconditioned set of linear inequality  
7       constraints;

8 applying term consistency to the set of preconditioned linear inequality  
9 constraints over the subbox  $\mathbf{X}$ , and  
10 excluding any portion of the subbox  $\mathbf{X}$  that violates any member of the set  
11 of preconditioned linear inequality constraints.

1 3. The method of claim 2, further comprising:  
2 keeping track of a least upper bound  $f\_bar$  of the function  $f(\mathbf{x})$  at a feasible  
3 point  $\mathbf{x}$  wherein  $p_i(\mathbf{x}) \leq 0$  ( $i=1, \dots, m$ ); and  
4 including  $f(\mathbf{x}) \leq f\_bar$  in the set of inequality constraints prior to  
5 linearizing the set of inequality constraints.

1 4. The method of claim 2, further comprising removing from  
2 consideration any inequality constraints that are not violated by more than a  
3 specified amount for purposes of applying term consistency prior to linearizing  
4 the set of inequality constraints.

1 5. The method of claim 1, wherein performing the interval global  
2 optimization process involves:  
3 keeping track of a least upper bound  $f\_bar$  of the function  $f(\mathbf{x})$  at a feasible  
4 point  $\mathbf{x}$ ;  
5 removing from consideration any subbox for which  $f(\mathbf{x}) > f\_bar$ ;  
6 applying term consistency to the  $f\_bar$  inequality  $f(\mathbf{x}) \leq f\_bar$  over the  
7 subbox  $\mathbf{X}$ ; and  
8 excluding any portion of the subbox  $\mathbf{X}$  that violates the  $f\_bar$  inequality.

1           6.       The method of claim 1, wherein if the subbox  $\mathbf{X}$  is strictly feasible  
2    ( $p_i(\mathbf{X}) < 0$  for all  $i=1, \dots, n$ ), performing the interval global optimization process  
3    involves:  
4           determining a gradient  $\mathbf{g}(\mathbf{x})$  of the function  $f(\mathbf{x})$ , wherein  $\mathbf{g}(\mathbf{x})$  includes  
5    components  $g_i(\mathbf{x})$  ( $i=1, \dots, n$ );  
6           removing from consideration any subbox for which  $\mathbf{g}(\mathbf{x})$  is bounded away  
7    from zero, thereby indicating that the subbox does not include an extremum of  
8     $f(\mathbf{x})$ ; and  
9           applying term consistency to each component  $g_i(\mathbf{x})=0$  ( $i=1, \dots, n$ ) of  $\mathbf{g}(\mathbf{x})=\mathbf{0}$   
10   over the subbox  $\mathbf{X}$ ; and  
11           excluding any portion of the subbox  $\mathbf{X}$  that violates any component of  
12    $\mathbf{g}(\mathbf{x})=\mathbf{0}$ .

1           7.       The method of claim 1, wherein if the subbox  $\mathbf{X}$  is strictly feasible  
2    ( $p_i(\mathbf{X}) < 0$  for all  $i=1, \dots, n$ ), performing the interval global optimization process  
3    involves:  
4           determining diagonal elements  $H_{ii}(\mathbf{x})$  ( $i=1, \dots, n$ ) of the Hessian of the  
5    function  $f(\mathbf{x})$ ;  
6           removing from consideration any subbox for which  $H_{ii}(\mathbf{x})$  a diagonal  
7    element of the Hessian over the subbox  $\mathbf{X}$  is always negative, indicating that the  
8    function  $f$  is not convex over the subbox  $\mathbf{X}$  and consequently does not contain a  
9    global minimum within the subbox  $\mathbf{X}$ ;  
10           applying term consistency to each inequality  $H_{ii}(\mathbf{x}) \geq 0$  ( $i=1, \dots, n$ ) over the  
11   subbox  $\mathbf{X}$ ; and  
12           excluding any portion of the subbox  $\mathbf{X}$  that violates a Hessian inequality.

1           8.       The method of claim 1, wherein if the subbox  $\mathbf{X}$  is strictly feasible  
2     ( $p_i(\mathbf{X}) < 0$  for all  $i=1, \dots, n$ ), performing the interval global optimization process  
3     involves:  
4           performing the Newton method, wherein performing the Newton method  
5     involves,  
6                    computing the Jacobian  $\mathbf{J}(\mathbf{x}, \mathbf{X})$  of the gradient of the  
7           function  $f$  evaluated with respect to a point  $\mathbf{x}$  over the subbox  $\mathbf{X}$ ,  
8                    computing an approximate inverse  $\mathbf{B}$  of the center of  
9            $\mathbf{J}(\mathbf{x}, \mathbf{X})$ ,  
10                   using the approximate inverse  $\mathbf{B}$  to analytically determine  
11           the system  $\mathbf{B}\mathbf{g}(\mathbf{x})$ , wherein  $\mathbf{g}(\mathbf{x})$  is the gradient of the function  $f(\mathbf{x})$ ,  
12           and wherein  $\mathbf{g}(\mathbf{x})$  includes components  $g_i(\mathbf{x})$  ( $i=1, \dots, n$ );  
13           applying term consistency to each component  $(\mathbf{B}\mathbf{g}(\mathbf{x}))_i = 0$  ( $i=1, \dots, n$ ) for  
14   each variable  $x_i$  ( $i=1, \dots, n$ ) over the subbox  $\mathbf{X}$ ; and  
15           excluding any portion of the subbox  $\mathbf{X}$  that violates a component.

1           9.       The method of claim 1, wherein applying term consistency  
2     involves:  
3           symbolically manipulating an equation within the computer system to  
4     solve for a term,  $g(x'_j)$ , thereby producing a modified equation  $g(x'_j) = h(\mathbf{x})$ ,  
5     wherein the term  $g(x'_j)$  can be analytically inverted to produce an inverse function  
6      $g^{-1}(\mathbf{y})$ ;  
7           substituting the subbox  $\mathbf{X}$  into the modified equation to produce the  
8     equation  $g(\mathbf{X}'_j) = h(\mathbf{X})$ ;  
9           solving for  $\mathbf{X}'_j = g^{-1}(h(\mathbf{X}))$ ; and  
10           intersecting  $\mathbf{X}'_j$  with the  $j$ -th element of the subbox  $\mathbf{X}$  to produce a new  
11   subbox  $\mathbf{X}^+$ ;

12            wherein the new subbox  $\mathbf{X}^+$  contains all solutions of the equation within  
13            the subbox  $\mathbf{X}$ , and wherein the size of the new subbox  $\mathbf{X}^+$  is less than or equal to  
14            the size of the subbox  $\mathbf{X}$ .

1            10.     The method of claim 1, further comprising performing the Newton  
2            method on the John conditions.

1            11.     A computer-readable storage medium storing instructions that  
2            when executed by a computer cause the computer to perform a method for using a  
3            computer system to solve a global inequality constrained optimization problem  
4            specified by a function  $f$  and a set of inequality constraints  $p_i(\mathbf{x}) \leq 0$  ( $i=1, \dots, m$ ),  
5            wherein  $f$  is a scalar function of a vector  $\mathbf{x} = (x_1, x_2, x_3, \dots, x_n)$ , the method  
6            comprising:

7            receiving a representation of the function  $f$  and the set of inequality  
8            constraints at the computer system;  
9            storing the representation in a memory within the computer system;  
10           performing an interval inequality constrained global optimization process  
11           to compute guaranteed bounds on a globally minimum value of the function  $f(\mathbf{x})$   
12           subject to the set of inequality constraints;

13           wherein performing the interval global optimization process involves,  
14                      applying term consistency to the set of inequality  
15                      constraints over a subbox  $\mathbf{X}$ , and  
16                      excluding any portion of the subbox  $\mathbf{X}$  that is proved to be  
17                      in violation of at least one member of the set of inequality  
18                      constraints.

1           12.     The computer-readable storage medium of claim 11, wherein the  
2     method further comprises:  
3           linearizing the set of inequality constraints to produce a set of linear  
4     inequality constraints with interval coefficients that enclose the nonlinear  
5     constraints;  
6           preconditioning the set of linear inequality constraints through additive  
7     linear combinations to produce a preconditioned set of linear inequality  
8     constraints;  
9           applying term consistency to the set of preconditioned linear inequality  
10    constraints over the subbox **X**, and  
11           excluding any portion of the subbox **X** that violates any member of the set  
12    of preconditioned linear inequality constraints.

1           13.     The computer-readable storage medium of claim 12, wherein the  
2     method further comprises:  
3           keeping track of a least upper bound  $f\_bar$  of the function  $f(\mathbf{x})$  at a feasible  
4     point  $\mathbf{x}$  wherein  $p_i(\mathbf{x}) \leq 0$  ( $i=1, \dots, m$ ); and  
5           including  $f(\mathbf{x}) \leq f\_bar$  in the set of inequality constraints prior to  
6     linearizing the set of inequality constraints.

1           14.     The computer-readable storage medium of claim 12, wherein the  
2     method further comprises removing from consideration any inequality constraints  
3     that are not violated by more than a specified amount for purposes of applying  
4     term consistency prior to linearizing the set of inequality constraints.

1           15.     The computer-readable storage medium of claim 11, wherein  
2     performing the interval global optimization process involves:

1 keeping track of a least upper bound  $f\_bar$  of the function  $f(\mathbf{x})$  at a feasible  
2 point  $\mathbf{x}$ ;  
3 removing from consideration any subbox for which  $f(\mathbf{x}) > f\_bar$ ;  
4 applying term consistency to the  $f\_bar$  inequality  $f(\mathbf{x}) \leq f\_bar$  over the  
5 subbox  $\mathbf{X}$ ; and  
6 excluding any portion of the subbox  $\mathbf{X}$  that violates the  $f\_bar$  inequality.

1 16. The computer-readable storage medium of claim 11, wherein if the  
2 subbox  $\mathbf{X}$  is strictly feasible ( $p_i(\mathbf{X}) < 0$  for all  $i=1, \dots, n$ ), performing the interval  
3 global optimization process involves:  
4 determining a gradient  $\mathbf{g}(\mathbf{x})$  of the function  $f(\mathbf{x})$ , wherein  $\mathbf{g}(\mathbf{x})$  includes  
5 components  $g_i(\mathbf{x})$  ( $i=1, \dots, n$ );  
6 removing from consideration any subbox for which  $\mathbf{g}(\mathbf{x})$  is bounded away  
7 from zero, thereby indicating that the subbox does not include an extremum of  
8  $f(\mathbf{x})$ ; and  
9 applying term consistency to each component  $g_i(\mathbf{x})=0$  ( $i=1, \dots, n$ ) of  $\mathbf{g}(\mathbf{x})=\mathbf{0}$   
10 over the subbox  $\mathbf{X}$ ; and  
11 excluding any portion of the subbox  $\mathbf{X}$  that violates any component of  
12  $\mathbf{g}(\mathbf{x})=\mathbf{0}$ .

1 17. The computer-readable storage medium of claim 11, wherein if the  
2 subbox  $\mathbf{X}$  is strictly feasible ( $p_i(\mathbf{X}) < 0$  for all  $i=1, \dots, n$ ), performing the interval  
3 global optimization process involves:  
4 determining diagonal elements  $H_{ii}(\mathbf{x})$  ( $i=1, \dots, n$ ) of the Hessian of the  
5 function  $f(\mathbf{x})$ ;  
6 removing from consideration any subbox for which  $H_{ii}(\mathbf{x})$  a diagonal  
7 element of the Hessian over the subbox  $\mathbf{X}$  is always negative, indicating that the

8 function  $f$  is not convex over the subbox  $\mathbf{X}$  and consequently does not contain a  
 9 global minimum within the subbox  $\mathbf{X}$ ;  
 10 applying term consistency to each inequality  $H_{ii}(\mathbf{x}) \geq 0$  ( $i=1, \dots, n$ ) over the  
 11 subbox  $\mathbf{X}$ ; and  
 12 excluding any portion of the subbox  $\mathbf{X}$  that violates a Hessian inequality.

1 18. The computer-readable storage medium of claim 11, wherein if the  
 2 subbox  $\mathbf{X}$  is strictly feasible ( $p_i(\mathbf{X}) < 0$  for all  $i=1, \dots, n$ ), performing the interval  
 3 global optimization process involves:

4 performing the Newton method, wherein performing the Newton method  
 5 involves,

6 computing the Jacobian  $\mathbf{J}(\mathbf{x}, \mathbf{X})$  of the gradient of the  
 7 function  $f$  evaluated with respect to a point  $\mathbf{x}$  over the subbox  $\mathbf{X}$ ,  
 8 computing an approximate inverse  $\mathbf{B}$  of the center of  
 9  $\mathbf{J}(\mathbf{x}, \mathbf{X})$ ,  
 10 using the approximate inverse  $\mathbf{B}$  to analytically determine  
 11 the system  $\mathbf{B}\mathbf{g}(\mathbf{x})$ , wherein  $\mathbf{g}(\mathbf{x})$  is the gradient of the function  $f(\mathbf{x})$ ,  
 12 and wherein  $\mathbf{g}(\mathbf{x})$  includes components  $g_i(\mathbf{x})$  ( $i=1, \dots, n$ );

13 applying term consistency to each component  $(\mathbf{B}\mathbf{g}(\mathbf{x}))_i = 0$  ( $i=1, \dots, n$ ) for  
 14 each variable  $x_i$  ( $i=1, \dots, n$ ) over the subbox  $\mathbf{X}$ ; and  
 15 excluding any portion of the subbox  $\mathbf{X}$  that violates a component.

1 19. The computer-readable storage medium of claim 11, wherein  
 2 applying term consistency involves:  
 3 symbolically manipulating an equation within the computer system to  
 4 solve for a term,  $g(x'_j)$ , thereby producing a modified equation  $g(x'_j) = h(\mathbf{x})$ ,



5 wherein the term  $g(x'_j)$  can be analytically inverted to produce an inverse function  
 6  $g^{-1}(y)$ ;  
 7 substituting the subbox  $\mathbf{X}$  into the modified equation to produce the  
 8 equation  $g(x'_j) = h(\mathbf{X})$ ;  
 9 solving for  $x'_j = g^{-1}(h(\mathbf{X}))$ ; and  
 10 intersecting  $x'_j$  with the  $j$ -th element of the subbox  $\mathbf{X}$  to produce a new  
 11 subbox  $\mathbf{X}^+$ ;  
 12 wherein the new subbox  $\mathbf{X}^+$  contains all solutions of the equation within  
 13 the subbox  $\mathbf{X}$ , and wherein the size of the new subbox  $\mathbf{X}^+$  is less than or equal to  
 14 the size of the subbox  $\mathbf{X}$ .

1 20. The computer-readable storage medium of claim 11, wherein the  
 2 method further comprises performing the Newton method on the John conditions.

1 21. An apparatus for using a computer system to solve a global  
 2 inequality constrained optimization problem specified by a function  $f$  and a set of  
 3 inequality constraints  $p_i(\mathbf{x}) \leq 0$  ( $i=1, \dots, m$ ), wherein  $f$  is a scalar function of a  
 4 vector  $\mathbf{x} = (x_1, x_2, x_3, \dots, x_n)$ , the apparatus comprising:  
 5 a receiving mechanism that is configured to receive a representation of the  
 6 function  $f$  and the set of inequality constraints at the computer system;  
 7 a memory within the computer system for storing the representation;  
 8 a global optimizer that is configured to perform an interval inequality  
 9 constrained global optimization process to compute guaranteed bounds on a  
 10 globally minimum value of the function  $f(\mathbf{x})$  subject to the set of inequality  
 11 constraints;  
 12 a term consistency mechanism within the global optimizer that is  
 13 configured to,

14                    apply term consistency to the set of inequality constraints  
15                    over a subbox  $\mathbf{X}$ , and to  
16                    exclude any portion of the subbox  $\mathbf{X}$  that is proved to be in  
17                    violation of at least one member of the set of inequality constraints.

1            22.    The apparatus of claim 21, further comprising:  
2            a linearizing mechanism that is configured to linearize the set of inequality  
3 constraints to produce a set of linear inequality constraints with interval  
4 coefficients that enclose the nonlinear constraints; and  
5            a preconditioning mechanism that is configured to precondition the set of  
6 linear inequality constraints through additive linear combinations to produce a  
7 preconditioned set of linear inequality constraints;  
8            wherein the term consistency mechanism is configured to,  
9            apply term consistency to the set of preconditioned linear  
10            inequality constraints over the subbox  $\mathbf{X}$ , and to  
11            exclude any portion of the subbox  $\mathbf{X}$  that violates any  
12            member of the set of preconditioned linear inequality constraints.

1            23.    The apparatus of claim 22, wherein the global optimizer is  
2 configured to:  
3            keep track of a least upper bound  $f\_bar$  of the function  $f(\mathbf{x})$  at a feasible  
4 point  $\mathbf{x}$  wherein  $p_i(\mathbf{x}) \leq 0$  ( $i=1, \dots, m$ ); and to  
5            include  $f(\mathbf{x}) \leq f\_bar$  in the set of inequality constraints prior to linearizing  
6 the set of inequality constraints.

1            24.    The apparatus of claim 22, wherein the term consistency  
2 mechanism is configured to remove from consideration any inequality constraints

3 that are not violated by more than a specified amount for purposes of applying  
4 term consistency prior to linearizing the set of inequality constraints.

1 25. The apparatus of claim 21,  
2 wherein the global optimizer is configured to,  
3 keep track of a least upper bound  $f\_bar$  of the function  $f(\mathbf{x})$   
4 at a feasible point  $\mathbf{x}$ , and to  
5 remove from consideration any subbox for which  
6  $f(\mathbf{x}) > f\_bar$ ;  
7 wherein the term consistency mechanism is configured to,  
8 apply term consistency to the  $f\_bar$  inequality  $f(\mathbf{x}) \leq f\_bar$   
9 over the subbox  $\mathbf{X}$ , and to  
10 exclude any portion of the subbox  $\mathbf{X}$  that violates the  $f\_bar$   
11 inequality.

1 26. The apparatus of claim 21, wherein if the subbox  $\mathbf{X}$  is strictly  
2 feasible ( $p_i(\mathbf{X}) < 0$  for all  $i=1, \dots, n$ ):  
3 the global optimizer is configured to,  
4 determine a gradient  $\mathbf{g}(\mathbf{x})$  of the function  $f(\mathbf{x})$ , wherein  $\mathbf{g}(\mathbf{x})$   
5 includes components  $g_i(\mathbf{x})$  ( $i=1, \dots, n$ ), and to  
6 remove from consideration any subbox for which  $\mathbf{g}(\mathbf{x})$  is  
7 bounded away from zero, thereby indicating that the subbox does  
8 not include an extremum of  $f(\mathbf{x})$ ; and  
9 the term consistency mechanism is configured to,  
10 apply term consistency to each component  $g_i(\mathbf{x})=0$   
11 ( $i=1, \dots, n$ ) of  $\mathbf{g}(\mathbf{x})=\mathbf{0}$  over the subbox  $\mathbf{X}$ , and to

12                   exclude any portion of the subbox **X** that violates any  
13                   component of  $\mathbf{g}(\mathbf{x})=0$ .

1           27.    The apparatus of claim 21, wherein if the subbox **X** is strictly  
2   feasible ( $p_i(\mathbf{X}) < 0$  for all  $i=1, \dots, n$ ):  
3           the global optimizer is configured to,  
4                   determine diagonal elements  $H_{ii}(\mathbf{x})$  ( $i=1, \dots, n$ ) of the  
5                   Hessian of the function  $f(\mathbf{x})$ , and to  
6                   remove from consideration any subbox for which  $H_{ii}(\mathbf{x})$  a  
7                   diagonal element of the Hessian over the subbox **X** is always  
8                   negative, indicating that the function  $f$  is not convex over the  
9                   subbox **X** and consequently does not contain a global minimum  
10                  within the subbox **X**; and  
11          the term consistency mechanism is configured to,  
12                  apply term consistency to each inequality  $H_{ii}(\mathbf{x}) \geq 0$   
13                  ( $i=1, \dots, n$ ) over the subbox **X**, and to  
14                  exclude any portion of the subbox **X** that violates a Hessian  
15                  inequality.

1           28.    The apparatus of claim 21, wherein if the subbox **X** is strictly  
2   feasible ( $p_i(\mathbf{X}) < 0$  for all  $i=1, \dots, n$ ):  
3           the global optimizer is configured to perform the Newton method, wherein  
4   performing the Newton method involves,  
5                   computing the Jacobian  $\mathbf{J}(\mathbf{x}, \mathbf{X})$  of the gradient of the  
6                   function  $f$  evaluated with respect to a point  $\mathbf{x}$  over the subbox **X**,  
7                   computing an approximate inverse  $\mathbf{B}$  of the center of  
8                    $\mathbf{J}(\mathbf{x}, \mathbf{X})$ , and

9 using the approximate inverse  $\mathbf{B}$  to analytically determine  
 10 the system  $\mathbf{B}\mathbf{g}(\mathbf{x})$ , wherein  $\mathbf{g}(\mathbf{x})$  is the gradient of the function  $f(\mathbf{x})$ ,  
 11 and wherein  $\mathbf{g}(\mathbf{x})$  includes components  $g_i(\mathbf{x})$  ( $i=1, \dots, n$ ); and  
 12 the term consistency mechanism is configured to,  
 13 apply term consistency to each component  $(\mathbf{B}\mathbf{g}(\mathbf{x}))_i = 0$   
 14 ( $i=1, \dots, n$ ) for each variable  $x_i$  ( $i=1, \dots, n$ ) over the subbox  $\mathbf{X}$ , and to  
 15 exclude any portion of the subbox  $\mathbf{X}$  that violates a  
 16 component.

1 29. The apparatus of claim 21, wherein the term consistency  
 2 mechanism is configured to:  
 3 symbolically manipulate an equation within the computer system to solve  
 4 for a term,  $g(x'_j)$ , thereby producing a modified equation  $g(x'_j) = h(\mathbf{x})$ , wherein  
 5 the term  $g(x'_j)$  can be analytically inverted to produce an inverse function  $g^{-1}(y)$ ;  
 6 substitute the subbox  $\mathbf{X}$  into the modified equation to produce the equation  
 7  $g(\mathbf{X}'_j) = h(\mathbf{X})$ ;  
 8 solve for  $\mathbf{X}'_j = g^{-1}(h(\mathbf{X}))$ ; and  
 9 intersect  $\mathbf{X}'_j$  with the  $j$ -th element of the subbox  $\mathbf{X}$  to produce a new  
 10 subbox  $\mathbf{X}^+$ ;  
 11 wherein the new subbox  $\mathbf{X}^+$  contains all solutions of the equation within  
 12 the subbox  $\mathbf{X}$ , and wherein the size of the new subbox  $\mathbf{X}^+$  is less than or equal to  
 13 the size of the subbox  $\mathbf{X}$ .

1 30. The apparatus of claim 21, wherein the global optimizer is  
 2 configured to apply the Newton method to the John conditions.